On the existence of weak solutions to the three-dimensional steady compressible Navier-Stokes equations in bounded domains

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Abstract

We prove the existence of a weak solution to the three-dimensional steady compressible isentropic Navier-Stokes equations in bounded domains for any specific heat ratio $\gamma > 1$. Generally speaking, the proof is based on the new weighted estimates of both pressure and kinetic energy for the approximate system which result in some higher integrability of the density, and the method of weak convergence. Comparing with [12] where the spatially periodic case was studied, here we have to control the additional integral terms of both pressure and kinetic energy involving with the points near the boundary which become degenerate when the points approach the boundary. Such integral terms are estimated using some new techniques, i.e., we use the techniques of the mirror image and boundary straightening to prove that the weighted estimates of both pressure and kinetic energy for the points near the boundary can be controlled by the weighted estimates for the points on the boundary. Moreover, we prove that once the weighted estimates of the kinetic energy in the direction of the unit normal to the boundary are bounded, we can control the weighted estimates of the total energy on the boundary.

Keywords: Steady compressible Navier-Stokes equations, existence for any $\gamma > 1$, weighted estimate, bounded domains, viscous flux.

1 Introduction

In this paper we shall prove the existence of a weak solution $(\rho, \mathbf{u})$ to the following steady isentropic compressible Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^3$ for any specific heat ratio $\gamma > 1$:

$$\operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$-\mu\triangle\mathbf{u} - \tilde{\mu}\nabla\operatorname{div}\mathbf{u} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \rho\mathbf{f} + \mathbf{g} \tag{1.2}$$

with Dirichlet boundary condition

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega. \tag{1.3}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity and ρ is the density, the viscosity constants μ and $\tilde{\mu}$ satisfy $\mu > 0$, $\tilde{\mu} = \mu + \lambda$ with $\lambda + 2\mu/3 \geq 0$, the pressure P for isentropic flows is given by

$$P(\rho) = a\rho^\gamma$$

with a being a positive constant and $\gamma > 1$ the specific heat ratio, $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$ are the external forces, and for simplicity, we assume that

$$\mathbf{f},\ \mathbf{g} \in L^\infty(\Omega).$$

Moreover, the total mass is prescribed:

$$\int_\Omega \rho dx = M > 0.$$

In the last decades, the well-posedness of the equations (1.1), (1.2) for large $\mathbf{f}$ and $\mathbf{g}$ has been investigated by a number of researchers. In 1998, under the assumption that $\gamma > 1$ in two dimensions and $\gamma > 5/3$ in three dimensions, Lions [13] proved the existence of weak solutions to boundary value probelms for (1.1), (1.2). Roughly speaking, the condition on γ comes from the integrability of the density ρ in $L^p(\Omega)$. The higher integrability of ρ has, the smaller γ can be allowed. If $\mathbf{f}$ is potential, then weak solutions are shown to exist for any $\gamma > 3/2$, see [15]. Then, Frehse, Goj, Steinhauer, Plotnikov and Sokolowsk established an improved integrability bound for the density by deriving a new weighted estimate of the pressure in [5, 16], where the authors assumed a priori the L^1-boundedness of $\rho\mathbf{u}^2$ which, unfortunately, was not shown to hold. Later, by combining the L^∞-estimate of $\triangle^{-1}P$ with the (usual) energy and density bounds, Březina and Novotný [1] were able to show the existence of spatially periodic weak solutions to (1.1), (1.2) with some symmetries for any $\gamma > (3+\sqrt{41})/8$ when $\mathbf{f}$ is potential, or for any $\gamma > 1.53$ when $\mathbf{f} \in L^\infty$, without assuming the boundedness of $\rho\mathbf{u}^2$ in L^1. Recently, Frehse, Steinhauer and Weigant [8] established the existence of weak solutions to the Dirichlet problem for (1.1), (1.2) in three dimensions for any $\gamma > 4/3$ in the framework of [1]. Also, the existence of weak solutions in the two-dimensional isothermal case ($\gamma = 1$) was obtained in [6, 7].

More recently, the authors of this paper [12] proved the existence of a spatially periodic weak solution with the same symmetries as in [1] to the system (1.1), (1.2) for any specific heat ratio $\gamma > 1$. The proof is based on the weighted estimates of both pressure and kinetic energy for the approximate system which result in some higher integrability of the density, and the method of weak convergence.

The aim of this paper is to prove the existence of a weak solution to (1.1)–(1.3) in bounded domains in $\mathbb{R}^3$ for any specific heat ratio $\gamma > 1$. Comparing with [12], here we have to establish the estimates involving with the points near the boundary which are degenerate. So, we cannot use the same techniques as in the derivation of the estimates of the interior points in [12] to get the weighted estimates of both pressure and kinetic energy for the points near the boundary. Instead, such weighted estimates are established using some new test functions, see the paragraph below Theorem 1.2 for more details on the proof idea.

We mention that for a 3D model of steady compressible heat-conducting flows (i.e., the steady compressible Navier-Stokes-Fourier system), Mucha and Pokorný [14] recently studied the existence of weak solutions under some assumptions on the pressure and heat-conductivity, which unfortunately excludes the case of polytropic idea gases. For the corresponding non-steady system to (1.1), (1.2) with large initial data, Lions [13] first proved the global existence of weak solutions in the case of $\gamma \geq 3n/(n+2)$ ($n = 2, 3$: dimension). His result has been improved and generalized recently in [2, 3, 4, 10, 11] and among others, where the condition $\gamma > 3/2$ is required in three dimensions for general initial data.

Before defining a weak solution to (1.1)–(1.3), we introduce the notation used throughout this paper.

Notation: Let G be a domain in $\mathbb{R}^3$. We denote by $L^p(G)$ the Lebesgue spaces with norm $\|\cdot\|_{L^p}$, by $W^{k,p}(G)$ ($k \in \mathbb{N}$) the Sobolev spaces with norm $\|\cdot\|_{W^{m,p}}$, by $C^k(G)$ (resp.

$C^k(\overline{G})$) the space of kth-times continuously differentiable functions in G (resp. $\overline{G}$). In particular, $H^m(G) \equiv W^{m,2}(G)$ with $\|\cdot\|_{H^m}$. By $\mathcal{D}'(G)$ we denote the dual space of $\mathcal{D}(G)$. $B_R(a) := \{x \in \mathbb{R}^3 : \ |x-a| < R\}$ denotes the open ball centered at a with radius R. The capital letter C (sometimes used as $C(X, Y, \cdots)$ to emphasize the dependence on $X, Y, \cdots$) denotes a generic positive constant which can vary from line to line.

Now, let us recall the definition of a bounded energy weak solution to (1.1)–(1.3).

Definition 1.1. (Renormalized bounded energy weak solution) *We call $(\rho, \mathbf{u})$ a renormalized bounded energy weak solution to the system (1.1)–(1.3), if the following is satisfied.*

(1) $\rho \geq 0$, $\rho \in L^\gamma(\Omega)$, $\mathbf{u} \in H_0^1(\Omega)$, $\int_\Omega \rho(x)dx = M > 0$.

(2) $(\rho, \mathbf{u})$ satisfies the energy inequality:

$$\int_\Omega \left(\mu|\nabla \mathbf{u}|^2 + \tilde{\mu}|\mathrm{div}\, \mathbf{u}|^2\right)dx \leq \int_\Omega (\rho \mathbf{f} + \mathbf{g}) \cdot \mathbf{u} dx. \tag{1.4}$$

(3) The system (1.1), (1.2) holds in the sense of $\mathcal{D}'(\Omega)$.

(4) The mass equation (1.1) holds in the sense of renormalized solutions, i.e.,

$$\mathrm{div}[b(\rho)\mathbf{u}] + [b'(\rho)\rho - b(\rho)]\mathrm{div}\mathbf{u} = 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{1.5}$$

for any $b \in C^1(\mathbb{R})$, such that $b'(z) = 0$ when z is big enough.

Then, the main result of the current paper reads as

Theorem 1.2. *Let Ω be a C^2-domain in $\mathbb{R}^3$, $\mathbf{f}, \mathbf{g} \in L^\infty(\Omega)$. Then for any $\gamma > 1$, there exists a renormalized bounded energy weak solution $(\rho, \mathbf{u})$ to the system (1.1)–(1.3).*

Roughly speaking, the proof of Theorem 1.2 is based on the uniform a priori estimates for the approximate solutions and the weak convergence method in the framework of Lions [13]. Comparing with [12] where the spatially periodic case was studied, here we have to control the additional integral terms of both pressure and kinetic energy involving with the points near the boundary which become degenerate when the points approach the boundary. Such integral terms are estimated using some new techniques, including the techniques of the mirror image and boundary straightening. The basic steps of the proof are the following: First, we use the distance function as a test function to get the weighted estimates of both pressure and kinetic energy in the direction of the unit normal to $\partial\Omega$, i.e. the weighted estimates of $P_\varepsilon + \rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \nabla d(x))^2$ for the points on $\partial\Omega$, where $\nabla d(x)$ is the normal direction of the boundary at x_0 satisfying $d(x) = |x - x_0|$. Second, by straightening the boundary locally, we can show that the weighted estimates of the total kinetic energy for the points on $\partial\Omega$ can be controlled by exploiting the weighted estimates of $\rho_\varepsilon(\mathbf{u}_\varepsilon \cdot \nabla d(x))^2$. Then, by using the method of the mirror image which is often used to construct the Green's function of the first kind for a ball or a half-ball, etc, we can construct a special test function to prove that the weighted estimates of both pressure and kinetic energy for the points near the boundary can be controlled by the weighted estimates of the points on the boundary $\partial\Omega$, and thus the weighted estimates for both pressure and the total kinetic energy can be closed, leading to the desired a priori estimates, in particular, the higher integrability of the density. Finally, with the help of the established a priori estimates, we can take to the limit in the same manner as in [12] to complete the proof of Theorem 1.2.

This paper is organized as follows. In Section 2, we first construct a sequence of approximate strong solutions $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ and then in Subsection 2.2 we derive the uniform weighted estimates for both pressure P_ε and kinetic energy $\rho_\varepsilon|\mathbf{u}_\varepsilon|^2$ in $\bar{\Omega}$. In Subsection 2.3 we take the first level limit $\varepsilon \to 0$ and $\varepsilon_0 \to 0$ and then show the additional uniform estimates for the velocity $\mathbf{u}_\delta$ and the pressure P_δ in terms of the quantity $A = \|P_\delta^\alpha|\mathbf{u}_\delta|^2 + \rho_\delta^\alpha|\mathbf{u}_\delta|^{2+2\alpha}\|_{L^1}$,

$0 < \alpha < 1$. Then, by using a bootstrap argument, we prove that A is uniformly bounded which in turn implies the uniform H^1-boundedness of $\mathbf{u}_\delta$, and the L^r-boundedness of P_δ, $\rho_\delta \mathbf{u}_\delta$ and $\rho_\delta |\mathbf{u}_\delta|^2$ for some $r > 1$. In Section 3, we prove the main theorem by using the weak convergence method in the framework of Lions [13].

2 Uniform estimates of the approximate solutions

2.1 The approximate system

To prove Theorem 1.2, we first work with the standard approximation problem in Ω with positive parameters ε, δ, $\varepsilon_0 < 1$:

$$\varepsilon_0(\rho - h) + \operatorname{div}(\rho \mathbf{u}) - \varepsilon \Delta \rho = 0, \tag{2.6}$$

$$\varepsilon_0(\rho + h)\mathbf{u} - \mu \triangle \mathbf{u} - \tilde{\mu} \nabla \operatorname{div} \mathbf{u} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \varepsilon \nabla \rho \cdot \nabla \mathbf{u} + \nabla P_\delta = \rho \mathbf{f} + \mathbf{g} \tag{2.7}$$

with boundary conditions

$$\mathbf{u} = \partial_n \rho = 0 \qquad \text{on } \partial\Omega, \tag{2.8}$$

where $h = M|\Omega|^{-1}$, $P_\delta = \rho^\gamma + \delta \rho^4$, and n is the outer normal vector to $\partial\Omega$.

According to [15], we have the following existence result for the problem (2.6)–(2.8).

Proposition 2.1. *There is at least one strong solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ to the problem (2.6)–(2.8) with the following properties ($\overline{\gamma} = \max\{\gamma, 4\}$):*

$$\rho_\varepsilon \in H^2(\Omega),\ \mathbf{u}_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega), \quad \rho_\varepsilon \geq 0 \ a.e.\ in\ \Omega, \quad \int_\Omega \rho_\varepsilon dx = M; \tag{2.9}$$

$$\|\rho_\varepsilon\|_{H^2} + \|\mathbf{u}_\varepsilon\|_{H^2} \leq C(\varepsilon_0, \varepsilon, \delta, M, \mathbf{f}, \mathbf{g}, h); \tag{2.10}$$

$$\|\mathbf{u}_\varepsilon\|_{H^1} \leq C(\Omega, h, \mathbf{f}, \mathbf{g})(1 + \|\rho_\varepsilon\|_{L^8}), \tag{2.11}$$

$$\varepsilon \|\nabla \rho_\varepsilon\|_{L^2}^2 \leq C(\Omega, h, \mathbf{f}, \mathbf{g})(1 + \|\rho_\varepsilon\|_{L^8})^3, \tag{2.12}$$

$$\delta \|\rho_\varepsilon\|_{L^8} \leq C(\Omega, h, \mathbf{f}, \mathbf{g}). \tag{2.13}$$

Before passing to the limit $\varepsilon \to 0$, $\varepsilon_0 \to 0$, $\delta \to 0$ to get the existence of a weak solution to the system (1.1), (1.2), we need first to show necessary uniform weighted estimates for $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$. In fact, we will prove the following theorem:

Theorem 2.2. *Let $(\rho_\varepsilon, \boldsymbol{u}_\varepsilon)$ be the solution of the approximate problem (2.6)–(2.8) established in Proposition 2.1. Then, for any $\delta > 0$, there exists a constant $\delta_0 > 0$, such that for any $\varepsilon, \varepsilon_0 < \delta_0$,*

$$\int_\Omega \frac{P_{\delta_\varepsilon} + \rho_\varepsilon |\mathbf{u}_\varepsilon|^2}{|x - x_0|^\alpha} dx \leq C\big(1 + \|P_\varepsilon\|_{L^1} + \|\rho_\varepsilon |\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|_{H^1}^2\big) \tag{2.14}$$

for all $\alpha \in (0,1)$, $x_0 \in \bar{\Omega}$, where the constant C depends on $\mathbf{f}$, $\mathbf{g}$, μ, $\tilde{\mu}$, M, γ, Ω and α, but not on δ, ε, ε_0 and x_0.

The proof of Theorem 2.2 is broken up into several lemmas given in Subsection 2.2.

2.2 A potential estimate

In this subsection we will derive both the interior and boundary weighted estimates for P_ε and $\rho_\varepsilon |\mathbf{u}_\varepsilon|^2$ which can be understood as estimates in the Morrey space.

Lemma 2.3. *Let $(\rho_\varepsilon, \boldsymbol{u}_\varepsilon)$ be the solution of (2.6)–(2.8) given in Proposition 2.1. Then the following estimate holds.*

$$\int_{B_{R_0}(x_0)} \frac{P_{\delta_\varepsilon} + \rho_\varepsilon |\mathbf{u}_\varepsilon|^2}{|x-x_0|^\alpha} dx \le C\big(1 + \|P_\varepsilon\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|_{H^1}^2\big) \tag{2.15}$$

for all $\alpha \in (0,1)$, $x_0 \in \Omega$ and $R_0 = \frac{1}{3} dist(x_0, \partial\Omega)$, where the constant C depends on $\mathbf{f}$, $\mathbf{g}$, μ, R_0, $\tilde{\mu}$, M, γ and α, but not on δ, ε and ε_0.

Proof. For $x_0 \in \Omega$, we define $\phi = (\phi^1, \phi^2, \phi^3)$ with

$$\phi^i(x) = \frac{(x-x_0)^i}{|x-x_0|^\alpha} \eta(\frac{|x-x_0|}{R_0}), \quad i = 1,2,3,$$

where $0 < \alpha < 1$, $\eta \in C^\infty(0,\infty)$ is a cut-off function satisfying $0 \le \eta(t) \le 1$, $|D\eta| \le 2$ and

$$\eta(t) = \begin{cases} 1 & |t| \le 1, \\ 0 & |t| \ge 2. \end{cases}$$

Testing (2.8) by ϕ and performing a direct computation similar to that in [12], we obtain

$$\begin{aligned} &\int_{B_{R_0}(x_0)} \frac{P_{\delta_\varepsilon}}{|x-x_0|^\alpha} dx + (1-\alpha) \int_{B_{R_0}(x_0)} \frac{\rho_\varepsilon |\mathbf{u}_\varepsilon|^2}{|x-x_0|^\alpha} dx \\ &\le C\Big(1 + \|P_\varepsilon\|_{L^1(\Omega)} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|_{H^1}^2 + \varepsilon_0 \|\rho_\varepsilon \mathbf{u}_\varepsilon\|_{L^1} + \varepsilon^2 \|\nabla\rho_\varepsilon\|_{L^2}^2\Big). \end{aligned} \tag{2.16}$$

If we take $\varepsilon_0 < \delta$ and $\varepsilon < \delta^3$, then (2.15) follows from (2.12), (2.13) and (2.16) immediately. □

For $\sigma > 0$, let us set $\Gamma_\sigma = \{x \in \bar{\Omega} \mid d(x) < \sigma\}$. The following lemma relates the smoothness of the distance function d in Γ_σ to that of the boundary $\partial\Omega$ (we refer to the Appendix of Chapter 14 in [9] for the details).

Lemma 2.4. *Let Ω be bounded and $\partial\Omega \in C^k$ for $k \ge 2$. Then there exists a positive constant σ depending on Ω such that $d \in C^k(\Gamma_\sigma)$.*

Now, we are in a position to get the uniform weighted estimates for the points near the boundary. The following lemma is inspired by Lemma 3.3 in [8], but the difference lies in that we prove in addition a weighted estimate for the kinetic energy in the direction of the unit normal to $\partial\Omega$ for the points on $\partial\Omega$ which is crucial in our proof.

Lemma 2.5. *Let $(\rho_\varepsilon, \boldsymbol{u}_\varepsilon)$ be the solution of (2.6)–(2.8) given in Proposition 2.1, and $\sigma > 0$ be a constant such that $d(x) \in C^2(\Gamma_\sigma)$, Then, the following estimate holds*

$$\int_{B_{\frac{\sigma}{2}}(x_0)} \frac{P_\varepsilon |\nabla d(x)|^2 + \rho_\varepsilon (\mathbf{u}_\varepsilon \cdot \nabla d(x))^2}{|x-x_0|^\alpha} dx \le C\big(1 + \|P_\varepsilon\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|_{H^1}^2\big) \tag{2.17}$$

for all $\alpha \in (0,1)$ and $x_0 \in \partial\Omega$, where the constant C depends on $\mathbf{f}$, $\mathbf{g}$, μ, $\tilde{\mu}, M, \gamma$, σ and α, but not on δ, ε, ε_0 and x_0.

Proof. For $x_0 \in \partial\Omega$, $\alpha \in (0,1)$, $m = \frac{2}{2-\alpha}$, we define $\phi = (\phi^1, \phi^2, \phi^3)$ with

$$\phi^i(x) = \frac{d(x)\partial_i d(x)}{(d(x) + |x-x_0|^m)^\alpha} \eta\Big(\frac{2|x-x_0|}{\sigma}\Big) \quad \text{in } \Omega, \quad i = 1,2,3.$$

Multiplying (2.8) with ϕ and integrating over Ω, we find that

$$\int_{B_\sigma(x_0)\cap\Omega} P_\varepsilon \mathrm{div}\phi dx + \int_{B_\sigma(x_0)\cap\Omega} \rho_\varepsilon u_\varepsilon^i u_\varepsilon^j \partial_j \phi^i dx = \int_{B_\sigma(x_0)\cap\Omega} (\varepsilon_0(\rho_\varepsilon + h)\mathbf{u}_\varepsilon + \varepsilon\nabla\mathbf{u}_\varepsilon \cdot \nabla\rho_\varepsilon)\phi dx$$
$$+\mu \int_{B_\sigma(x_0)\cap\Omega} \nabla\mathbf{u}_\varepsilon : \nabla\phi dx + \tilde{\mu} \int_{B_\sigma(x_0)\cap\Omega} \mathrm{div}\mathbf{u}_\varepsilon \, \mathrm{div}\phi dx + \int_{B_\sigma(x_0)\cap\Omega} (\rho_\varepsilon \mathbf{f} + \mathbf{g})\phi dx. \quad (2.18)$$

A straightforward computation gives that

$$\begin{aligned}
\partial_j\phi^i(x) = & \frac{\partial_j d(x)\partial_i d(x) + d(x)\partial_{ij}d(x)}{(d(x)+|x-x_0|^m)^\alpha}\eta(\frac{2|x-x_0|}{\sigma}) \\
& -\alpha\frac{d(x)\partial_i d(x)[\partial_j d(x) + \partial_j|x-x_0|^m]}{(d(x)+|x-x_0|^m)^{\alpha+1}}\eta(|x-x_0|) + \frac{d(x)\partial_i d(x)}{(d(x)+|x-x_0|^m)^\alpha}\partial_j\eta(\frac{2|x-x_0|}{\sigma}) \\
= & \frac{d(x)\partial_{ij}d(x)}{(d(x)+|x-x_0|^m)^\alpha}\eta(\frac{2|x-x_0|}{\sigma}) + \frac{d(x)\partial_i d(x)}{(d(x)+|x-x_0|^m)^\alpha}\partial_j\eta(\frac{2|x-x_0|}{\sigma}) \\
& + \frac{(1-\alpha)d(x)+|x-x_0|^m}{2(d(x)+|x-x_0|^m)^{\alpha+1}}\partial_i d(x)\partial_j d(x)\eta(\frac{2|x-x_0|}{\sigma}) \\
& + \frac{(1-\alpha)d(x)+|x-x_0|^m}{2(d(x)+|x-x_0|^m)^{\alpha+1}}(\partial_i d(x) - M^i(x))(\partial_j d(x) - M^j(x))\eta(\frac{2|x-x_0|}{\sigma}) \\
& + \frac{2\alpha[\partial_j d(x)\partial_i|x-x_0|^m - \partial_i d(x)\partial_j|x-x_0|^m]}{2(d(x)+|x-x_0|^m)^{\alpha+1}}\eta(\frac{2|x-x_0|}{\sigma}) \\
& + \frac{\alpha^2 d^2(x)\partial_i|x-x_0|^m\partial_j|x-x_0|^m}{2(d(x)+|x-x_0|^m)^{\alpha+1}((1-\alpha)d(x)+|x-x_0|^m)}\eta(\frac{2|x-x_0|}{\sigma}),
\end{aligned} \quad (2.19)$$

where $M_i(x) = \alpha d(x)\partial_i|x-x_0|^m((1-\alpha)d(x)+|x-x_0|^m)^{-1}$, $i = 1,2,3$. From (2.19) it follows easily that

$$\mathrm{div}\phi(x) \geq \frac{(1-\alpha)|\nabla d(x)|^2}{2(d(x)+|x-x_0|^m)^\alpha}\eta(\frac{2|x-x_0|}{\sigma}) - C. \quad (2.20)$$

The advantage of the representation (2.19) lies in that one can clearly identify the symmetric and skew-symmetric parts. From (2.19) and (2.20) we get that for any $x_0 \in \partial\Omega$,

$$\int_{B_\sigma(x_0)\cap\Omega} P_{\delta_\varepsilon}\mathrm{div}\phi(x)dx \geq \int_{B_{\frac{\sigma}{2}}(x_0)\cap\Omega} \frac{(1-\alpha)P_{\delta_\varepsilon}|\nabla d(x)|^2}{2(d(x)+|x-x_0|^m)^\alpha} - C(\sigma)\|P_{\delta_\varepsilon}\|_{L^1} \quad (2.21)$$

and

$$\sum_{i,j=1}^3 \int_{B_\sigma(x_0)\cap\Omega} \rho_\varepsilon u_{i_\varepsilon} u_{j_\varepsilon}\partial_j\phi^i dx \geq \int_{B_{\frac{\sigma}{2}}(x_0)\cap\Omega} \frac{(1-\alpha)\rho_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla d(x))^2}{2(d(x)+|x-x_0|^m)^\alpha}$$
$$-C(\sigma)\|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1}. \quad (2.22)$$

Substituting (2.21) and (2.22) into (2.18), we obtain

$$\int_{B_{\frac{\sigma}{2}}(x_0)} \frac{P_\varepsilon|\nabla d(x)|^2 + \rho_\varepsilon(\mathbf{u}_\varepsilon\cdot\nabla d(x))^2}{|x-x_0|^\alpha}dx$$
$$\leq \frac{C}{1-\alpha}\big(1 + \|P_\varepsilon\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|_{H^1}^2 + \varepsilon_0\|\rho_\varepsilon\mathbf{u}_\varepsilon\|_{L^1} + \varepsilon^2\|\nabla\rho_\varepsilon\|_{L^2}^2\big). \quad (2.23)$$

Thus, taking $\varepsilon_0 < \delta$ and $\varepsilon < \delta^3$, we see that (2.17) follows from (2.12), (2.13) and (2.23) immediately. $\square$

For any $x_0 \in \partial\Omega$, since the Navier-Stokes system is invariant under the rotation transformation, we can assume that the x_3 coordinate axis lies in the direction $\nu(x_0)$, here $\nu(x_0)$ is the unit inner normal to $\partial\Omega$ at x_0. Consequently, one has $\nabla d(x_0) = \nu(x_0) = (0,0,1)$. Besides, by Lemma 2.4, $d \in C^2(\Gamma_\sigma)$, $\|d\|_{C^2(\Gamma_\sigma)} \le C$ and for any $\epsilon > 0$ small enough, whence, there exists a constant $\bar{\epsilon}_0 > 0$ such that for $\epsilon_0 = \min\{\bar{\epsilon}_0, \sigma\}$,

$$\|\nabla d(x) - (0,0,1)\|_{C^1(B(x_0,\epsilon_0)\cap\Gamma_\sigma)} < \epsilon,$$

which combined with Lemma 2.5 implies that for any $\bar{x}_0 \in B(x_0, \frac{\epsilon_0}{2}) \cap \partial\Omega$,

$$\int_{B(\bar{x}_0,\frac{\epsilon_0}{2})\cap\Gamma_\sigma} \frac{P_{\delta_\varepsilon} + \rho_\varepsilon u_{3_\varepsilon}}{|x-\bar{x}_0|^\alpha} dx \le C\big(1 + \|P_{\delta_\varepsilon}\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|^2_{H^1}\big) \tag{2.24}$$

Now, we define the following transformation φ which straightens the boundary $T = B(x_0,\epsilon_0)\cap\partial\Omega$ in $N = B(x_0,\epsilon_0)\cap\Gamma_\sigma$ to

$$y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = d(x_1,x_2,x_3),$$

and set

$$y_0 = \varphi(x_0), \quad N' = \varphi(N), \quad T' = \varphi(T), \quad \bar{\mathbf{u}}_\varepsilon(y) = \mathbf{u}_\varepsilon(x), \quad \bar{\rho}_\varepsilon(y) = \rho_\varepsilon(x).$$

Then, under this transformation, the system (2.6)–(2.8) is transformed to (here all derivatives are with respect to y unless noticed explicitly):

$$\varepsilon_0(\bar{\rho}_\varepsilon - h) + \operatorname{div}(\bar{\rho}_\varepsilon\bar{\mathbf{u}}_\varepsilon) - \varepsilon\Delta\bar{\rho}_\varepsilon = F_1(\bar{\rho}_\varepsilon, d, \bar{\mathbf{u}}_\varepsilon), \tag{2.25}$$

$$\begin{aligned}&\varepsilon_0(\bar{\rho}_\varepsilon + h)\bar{\mathbf{u}}_\varepsilon - \mu\triangle\bar{\mathbf{u}}_\varepsilon - \tilde{\mu}\nabla\operatorname{div}\bar{\mathbf{u}}_\varepsilon + \operatorname{div}(\bar{\rho}_\varepsilon\bar{\mathbf{u}}_\varepsilon\otimes\bar{\mathbf{u}}_\varepsilon) + \nabla P_{\delta_\varepsilon}(\bar{\rho}_\varepsilon) + \varepsilon\nabla\bar{\rho}_\varepsilon\cdot\nabla\bar{\mathbf{u}}_\varepsilon \\ &\quad = \bar{\rho}_\varepsilon\mathbf{f} + \mathbf{g} + \mathbf{F}_2(\bar{\rho}_\varepsilon, d, \bar{\mathbf{u}}_\varepsilon),\end{aligned} \tag{2.26}$$

where $\mathbf{F}_2 = (F_2^1, F_2^2, F_2^3)$ and

$$\begin{aligned}
F_1(\bar{\rho}_\varepsilon, d, \bar{\mathbf{u}}_\varepsilon) &= -(\partial_{x_i}d - \delta_{3i})\cdot\partial_{y_3}(\bar{\rho}_\varepsilon\bar{u}_{i_\varepsilon}) \\
&\quad -\varepsilon[2\partial_{x_1}d\partial_{13}\bar{\rho}_\varepsilon + 2\partial_{x_2}d\partial_{23}\bar{\rho}_\varepsilon + \partial_3\bar{\rho}_\varepsilon\Delta_x d + (|\nabla_x d|^2 - 1)\partial_3^2\bar{\rho}_\varepsilon], \\
F_2^j(\bar{\rho}_\varepsilon, d, \bar{\mathbf{u}}_\varepsilon) &= \mu[2\partial_{x_1}d\partial_{13}\bar{u}_{j_\varepsilon} + 2\partial_{x_2}d\partial_{23}\bar{u}_{j_\varepsilon} + \partial_3\bar{u}_{j_\varepsilon}\Delta_x d + (|\nabla_x d|^2 - 1)\partial_3^2\bar{u}_{j_\varepsilon}] \\
&\quad -\tilde{\mu}[(\partial_{x_j}d - \delta_{3j})\partial_3\operatorname{div}\bar{\mathbf{u}}_\varepsilon + \partial_{x_jx_i}d\partial_3\bar{u}_{i_\varepsilon} \\
&\quad +(\partial_{x_i}d - \delta_{3i})\cdot(\partial_{3j}\bar{u}_{i_\varepsilon} + (\partial_{x_j}d - \delta_{3j})\partial_3^2\bar{u}_{i_\varepsilon})] \\
&\quad -(\partial_{x_i}d - \delta_{3i})\cdot\{\partial_{y_3}(\bar{\rho}_\varepsilon\bar{u}_{i_\varepsilon}\bar{u}_{j_\varepsilon}) + \varepsilon[\partial_3\bar{\rho}_\varepsilon\partial_i\bar{u}_{j\varepsilon} + \partial_i\bar{\rho}_\varepsilon\partial_3\bar{u}^j + (\partial_{x_i}d - \delta_{3i})\partial_3\bar{\rho}_\varepsilon\partial_3\bar{u}_{j_\varepsilon}]\} \\
&\quad -(\partial_{x_j}d - \delta_{3j})\partial_3\bar{P}_{\delta_\varepsilon} \\
&=: O(\epsilon)F(\nabla^2\mathbf{u}_\varepsilon, \nabla\rho_\varepsilon\cdot\nabla\mathbf{u}_\varepsilon, \partial_3(\rho_\varepsilon u_{i_\varepsilon}u_{j_\varepsilon})) + O(|\nabla\mathbf{u}_\varepsilon|).
\end{aligned}$$

If we define $\tilde{\epsilon_0} := \frac{1}{1+\epsilon}\epsilon_0$, then (2.24) implies that for any $\bar{y}_0 \in B(y_0, \frac{\tilde{\epsilon}_0}{2}) \cap T'$, one has

$$\int_{B(\bar{y}_0,\frac{\tilde{\epsilon}_0}{2})\cap N'} \frac{\bar{P}_{\delta_\varepsilon} + \bar{\rho}_\varepsilon\bar{u}^2_{3_\varepsilon}}{|y-\bar{y}_0|^\alpha} dy \le C\big(1 + \|P_{\delta_\varepsilon}\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|^2_{H^1}\big). \tag{2.27}$$

Next, we show that with the help of the estimate (2.27), we can also control the weighted estimates of the total kinetic energy for the degenerate points on the boundary. In fact, one has the following lemma.

Lemma 2.6. *For any $\bar{y}_0 \in B(y_0, \frac{\tilde{\epsilon}_0}{2}) \cap T'$, we have*

$$\int_{B(\bar{y}_0,\frac{\tilde{\epsilon}_0}{2})\cap N'} \frac{\bar{\rho}_\varepsilon\bar{\mathbf{u}}^2_\varepsilon}{|y-\bar{y}_0|^\alpha} dy \le C\big(1 + \|P_{\delta_\varepsilon}\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|^2_{H^1}\big). \tag{2.28}$$

Proof. Set $\bar{N}' = \{(y_1, y_2, y_3) |\ (y_1, y_2, -y_3) \in N'\}$, $\tilde{N}' = N' \cup \bar{N}'$ and $y' = (y_1, y_2)$. Since $\bar{\rho}_\varepsilon$, $\bar{\mathbf{u}}_\varepsilon \in W^{2,2}(N')$ and $\mathbf{u}_\varepsilon = 0$ on T, we denote

$$\tilde{\mathbf{u}}_\varepsilon(y', y_3) = \begin{cases} -\bar{\mathbf{u}}_\varepsilon(y', -y_3) & \text{for } y_3 \le 0 \\ \ \bar{\mathbf{u}}_\varepsilon(y', y_3) & \text{for } y_3 \ge 0, \end{cases} \quad , \qquad \tilde{\rho_\varepsilon} = \begin{cases} \bar{\rho}_\varepsilon(y', -y_3) & \text{for } y_3 \le 0 \\ \ \bar{\rho}_\varepsilon(y', y_3) & \text{for } y_3 \ge 0. \end{cases}$$

to deduce that $\tilde{\mathbf{u}}_\varepsilon \in W^{2,2}(\tilde{N}')$, $\tilde{\rho_\varepsilon} \in W^{1,2}(\tilde{N}')$ and the following equations hold.

$$-\mathrm{div}(\tilde{\rho_\varepsilon}\tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) = G_\varepsilon, \qquad a.e. \ \text{ in } \tilde{N}' \tag{2.29}$$

where

$$G^i_\varepsilon(y', y_3) = \begin{cases} \big\{\varepsilon_0(\bar{\rho}_\varepsilon + h)\bar{\mathbf{u}}_\varepsilon - \mu\triangle\bar{\mathbf{u}}_\varepsilon - \bar{\mu}\nabla\mathrm{div}\bar{\mathbf{u}}_\varepsilon + \nabla\bar{P}_{\delta_\varepsilon} - \bar{\rho}_\varepsilon\bar{\mathbf{f}} + \varepsilon\nabla\bar{\rho}_\varepsilon\cdot\nabla\bar{\mathbf{u}}_\varepsilon - \bar{\mathbf{g}}\big\}(y', y_3) \\ -F^i_2(\bar{\rho}_\varepsilon, d, \bar{\mathbf{u}}_\varepsilon)(y', y_3) \qquad\qquad\qquad\qquad \text{for } y_3 \ge 0, \\ \big\{\varepsilon_0(\bar{\rho}_\varepsilon + h)\bar{\mathbf{u}}_\varepsilon - \mu\triangle\bar{\mathbf{u}}_\varepsilon - \bar{\mu}\nabla\mathrm{div}\bar{\mathbf{u}}_\varepsilon + \nabla\bar{P}_{\delta_\varepsilon} - \bar{\rho}_\varepsilon\bar{\mathbf{f}} + \varepsilon\nabla\bar{\rho}_\varepsilon\cdot\nabla\bar{\mathbf{u}}_\varepsilon - \bar{\mathbf{g}}\big\}(y', -y_3) \\ -F^i_2(\bar{\rho}_\varepsilon, d, \bar{\mathbf{u}}_\varepsilon)(y', -y_3) + 2\partial_3(\bar{\rho}_\varepsilon\bar{u}_{3_\varepsilon}\bar{u}_{j_\varepsilon})(y', -y_3) \qquad \text{for } y_3 < 0. \end{cases}$$

For any $\bar{y}_0 \in B(y_0, \frac{\tilde{\epsilon}_0}{2}) \cap T'$, multiplying (2.29) by $\phi^i(y) = \frac{(y-\bar{y}_0)^i}{|y-\bar{y}_0|^\alpha}\eta(\frac{2|y-\bar{y}_0|}{\tilde{\epsilon}_0})$ and integrating over $\tilde{N}'$, using the fact that $\phi^i(y', y_3) = \phi^i(y', -y_3) = \phi^i(y', 0)$ on T', we obtain by a direct calculation that

$$\begin{aligned}(1-\alpha)\int_{B(\bar{y}_0, \frac{\tilde{\epsilon}_0}{2})} \frac{P_{\delta_\varepsilon} + \bar{\rho}_\varepsilon\bar{\mathbf{u}}^2_\varepsilon}{|y-\bar{y}_0|^\alpha} dy &\le C(1 + \|P_{\delta_\varepsilon}\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|^2_{H^1}) \\ &+ \int_{B(\bar{y}_0, \frac{\tilde{\epsilon}_0}{2})} \frac{\bar{\rho}_\varepsilon|\bar{u}_{i_\varepsilon}\bar{u}_{3_\varepsilon}|}{|y-\bar{y}_0|^\alpha} dy + \varepsilon_0\|\rho_\varepsilon\mathbf{u}_\varepsilon\|_{L^1} + \varepsilon^2\|\nabla\bar{\rho}_\varepsilon\|^2_{L^2(N')}.\end{aligned} \tag{2.30}$$

If we take $\varepsilon_0 < \delta$ and $\varepsilon < \delta^3$, then (2.28) follows from (2.12), (2.13), (2.27), (2.30) and Young's inequality immediately. □

Then, by using the method of the mirror image which is often used to construct the Green's function of the first kind for a ball or a half-ball, etc., we can construct a special test function to prove that the weighted estimates of both pressure and kinetic energy for the degenerate points near the boundary can be controlled by the weighted estimates for the points on the boundary:

Lemma 2.7. *For any $\bar{y} \in B(y_0, \frac{\tilde{\epsilon}_0}{2}) \cap N'$, one has*

$$\int_{B(\bar{y}, \frac{\tilde{\epsilon}_0}{2}) \cap N'} \frac{\bar{P}_{\delta_\varepsilon} + \bar{\rho}_\varepsilon\bar{\mathbf{u}}^2_\varepsilon}{|y-\bar{y}|^\alpha} dy \le C(1 + \|P_{\delta_\varepsilon}\|_{L^1} + \|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1} + \|\mathbf{u}_\varepsilon\|^2_{H^1}). \tag{2.31}$$

Proof. For any $\bar{y} = (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in B(y_0, \frac{\tilde{\epsilon}_0}{2}) \cap N'$, let $\bar{y}^* = (\bar{y}_1, \bar{y}_2, -\bar{y}_3)$, $\bar{y}_0 = (\bar{y}_1, \bar{y}_2, 0)$, and define

$$\psi^i(y) = \Big[\frac{(y-\bar{y})^i}{|y-\bar{y}|^\alpha} - \frac{(1-2\delta_{3i})(y-\bar{y}^*)^i}{|y-\bar{y}^*|^\alpha}\Big]\eta(\frac{2|y-\bar{y}|}{\tilde{\epsilon}_0}), \quad i = 1, 2, 3,$$

then $\psi(y) = (\psi^1, \psi^2, \psi^2) = 0$ on $\partial(B(\bar{y}, \tilde{\epsilon}_0) \cap N')$.

Multiplying (2.26) with ψ and integrating over N', since $|y - \bar{y}_0| \le |y - \bar{y}^*|$ for all $y \in N'$ and

$$\begin{aligned}\partial_j\psi^i(y) = &\Big[\frac{(y-\bar{y})^i}{|y-\bar{y}|^\alpha} - \frac{(1-2\delta_{3i})(y-\bar{y}^*)^i}{|y-\bar{y}^*|^\alpha}\Big]\partial_j\eta(|y-\bar{y}|) \\ +&\Big\{\frac{\delta_{ij}}{|y-\bar{y}|^\alpha} - \alpha\frac{(y-\bar{y})^i(y-\bar{y})^j}{|y-\bar{y}|^{\alpha+2}} - \frac{(1-2\delta_{3i})\delta_{ij}}{|y-\bar{y}^*|^\alpha} + \alpha\frac{(1-2\delta_{3i})(y-\bar{y}^*)^i(y-\bar{y}^*)^j}{|y-\bar{y}^*|^{\alpha+2}}\Big\}\eta(|y-\bar{y}|),\end{aligned}$$

we find that

$$\int_{B(\bar{y},\frac{\tilde{\epsilon}_0}{2})\cap N'} \bar{P}_{\delta_\varepsilon}\mathrm{div}\psi dy \geq\ (3-\alpha)\int_{B(\bar{y},\frac{\tilde{\epsilon}_0}{2})\cap N'} \frac{\bar{P}_{\delta_\varepsilon}}{|y-\bar{y}|^\alpha}dy - C(\tilde{\epsilon}_0)\|P_{\delta_\varepsilon}\|_{L^1}$$
$$-C(\tilde{\epsilon}_0)\int_{B(\bar{y},\frac{\tilde{\epsilon}_0}{2})\cap N'} \frac{\bar{P}_{\delta_\varepsilon}}{|y-\bar{y}_0|^\alpha}dy \tag{2.32}$$

and

$$\sum_{i,j=1}^{3}\int_{B(\bar{y},\frac{\tilde{\epsilon}_0}{2})\cap N'} \bar{\rho}_\varepsilon \bar{u}^i_\varepsilon \bar{u}^j_\varepsilon \partial_j\psi^i dy \geq (1-\alpha)\int_{B(\bar{y},\frac{\tilde{\epsilon}_0}{4})\cap N'} \frac{\bar{\rho}_\varepsilon|\bar{\mathbf{u}}_\varepsilon|^2}{|y-\bar{y}|^\alpha}dy - C(\tilde{\epsilon}_0)\|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1}$$
$$-C(\tilde{\epsilon}_0)\int_{B(\bar{y},\frac{\tilde{\epsilon}_0}{2})\cap N'} \frac{\bar{\rho}_\varepsilon|\bar{\mathbf{u}}_\varepsilon|^2}{|y-\bar{y}_0|^\alpha}dy. \tag{2.33}$$

Thus, by (2.32), (2.33), Lemma 2.6 and a straightforward calculation, we infer that

$$\int_{B(\bar{y},\frac{\tilde{\epsilon}_0}{2})\cap N'} \frac{\bar{P}_{\delta_\varepsilon}+\bar{\rho}_\varepsilon\bar{\mathbf{u}}^2_\varepsilon}{|y-\bar{y}|^\alpha}dy \leq C(1+\|P_{\delta_\varepsilon}\|_{L^1}+\|\rho_\varepsilon|\mathbf{u}_\varepsilon|^2\|_{L^1}+\|\mathbf{u}_\varepsilon\|^2_{H^1})$$
$$+C(\epsilon)\int_{B(\bar{y}_0,\frac{\tilde{\epsilon}_0}{2})\cap N'} \frac{P_{\delta_\varepsilon}+\bar{\rho}_\varepsilon\bar{\mathbf{u}}^2_\varepsilon}{|y-\bar{y}|^\alpha}dy + \varepsilon_0\|\rho_\varepsilon\mathbf{u}_\varepsilon\|_{L^1} + \varepsilon^2\|\nabla\bar{\rho}_\varepsilon\|^2_{L^2(N')}. \tag{2.34}$$

If we take ϵ sufficiently small, $\varepsilon_0<\delta$ and $\varepsilon<\delta^3$, then (2.31) follows from (2.12), (2.13) and (2.34). □

Finally, taking $\delta_0=\delta^3$, we use Lemmas 2.3–2.7 and the finite covering theorem to obtain Theorem 2.2. This completes the proof of Theorem 2.2.

2.3 Vanishing limits as $\varepsilon\to 0$ and $\varepsilon_0\to 0$

According to [15] and Theorem 2.2, if we take the limits as $\varepsilon\to 0$ and $\varepsilon_0\to 0$ respectively, then there is at least a weak solution $(\rho_\delta,\mathbf{u}_\delta)$ to the problem

$$\mathrm{div}(\rho_\delta\mathbf{u}_\delta)=0, \tag{2.35}$$
$$-\mu\triangle\mathbf{u}_\delta-\tilde{\mu}\nabla\mathrm{div}\mathbf{u}_\delta+\mathrm{div}(\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta)+\nabla P_\delta(\rho_\delta)=\rho_\delta\mathbf{f}+\mathbf{g}, \tag{2.36}$$
$$\mathbf{u}_\delta|_{\partial\Omega}=0, \tag{2.37}$$

satisfying ($\overline{\gamma}=\max\{\gamma,4\}$):

$$\rho_\delta\in L^{2\overline{\gamma}}(\Omega),\quad \mathbf{u}_\delta\in H^1_0(\Omega),\quad \int_\Omega\rho_\delta dx=M; \tag{2.38}$$
$$\mathrm{div}[b(\rho_\delta)\mathbf{u}_\delta]+\big[b'(\rho_\delta)\rho_\delta-b(\rho_\delta)\big]\mathrm{div}\mathbf{u}_\delta=0 \quad \text{in } \mathcal{D}'(\Omega); \tag{2.39}$$
$$\int_\Omega\big[\mu|\nabla\mathbf{u}_\delta|^2+\tilde{\mu}|\mathrm{div}\mathbf{u}_\delta|^2\big]dx\leq\int_\Omega\rho_\delta\mathbf{f}\cdot\mathbf{u}_\delta+\mathbf{g}\cdot\mathbf{u}_\delta dx, \tag{2.40}$$
$$\int_\Omega\frac{P_\delta+\rho_\delta|\mathbf{u}_\delta|^2}{|x-x_0|^\alpha}dx\leq C\big(1+\|P_\delta\|_{L^1}+\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1}+\|\mathbf{u}_\delta\|^2_{H^1}\big) \tag{2.41}$$

for all $\alpha\in(0,1)$, $x_0\in\overline{\Omega}$, where the constant C depends on $\mathbf{f}$, $\mathbf{g}$, μ, $\tilde{\mu}$, M, γ, Ω and α, but not on δ and x_0, b is the same as in (1.5).

Now, with the help of (2.41) we are able to derive the uniform-in-δ estimates for (ρ_δ,u_δ), which will be used in passing to the limit as $\delta\to 0$ in the next section, to get a weak solution of the system (1.1), (1.2). More precisely, denoting

$$A=\|P^\alpha_\delta|\mathbf{u}_\delta|^2+\rho^\alpha_\delta|\mathbf{u}_\delta|^{2+2\alpha}\|_{L^1},\qquad 0<\alpha<1, \tag{2.42}$$

we have the following uniform estimates:

Theorem 2.8. *For A defined by (2.42), it holds for any $1 < r < 2 - 1/\gamma$ that*

$$A + \|\mathbf{u}_\delta\|_{H^1} + \|P_\delta\|_{L^r} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^r} + \|\rho_\delta \mathbf{u}_\delta\|_{L^r} \leq C, \tag{2.43}$$

$$\delta \int_\Omega \rho_\delta^6 dx \to 0 \qquad \text{as } \delta \to 0, \tag{2.44}$$

where the constant C depends only on $\|\mathbf{f}\|_{L^\infty}$, $\|\mathbf{g}\|_{L^\infty}$, μ, $\tilde{\mu}$, M, γ and α (but not on δ).

Theorem 2.8 can be obtained by arguments very similar to those used in [12], and hence we omit the details of the proof. Instead, we only briefly describe the main steps of the proof here.

Lemma 2.9. *Let $(\rho_\delta, \mathbf{u}_\delta)$ be the solution of the approximate problem (2.35)–(2.37). Then,*

$$\|\mathbf{u}_\delta\|_{H^1} \leq C A^{\frac{\gamma-1}{4(\alpha\gamma+\gamma-2)}},$$

where the constant C depends on $\|\mathbf{f}\|_{L^\infty}$, $\|\mathbf{g}\|_{L^\infty}$, μ, $\tilde{\mu}$, M, γ and Ω, but not on δ.

Lemma 2.10. *Let $(\rho_\delta, \mathbf{u}_\delta)$ be the solution of (2.35)–(2.37). Then for $s \in (1, \alpha + 1 - \alpha/\gamma]$, we have*

$$\|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^s}^s + \|P_\delta\|_{L^s}^s \leq C\Big(1 + A^{\frac{\gamma s - 1}{\gamma\alpha+\gamma-2}}\Big),$$

where the constant C depends only on $\|\mathbf{f}\|_{L^\infty}$, $\|\mathbf{g}\|_{L^\infty}$, μ, λ, M, γ and Ω, but not on δ.

Lemma 2.11. *Let A be defined by (2.42), then we have*

$$A \leq C\|\mathbf{u}_\delta\|_{H^1}^2 (1 + \|P_\delta\|_{L^1} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1} + \|\mathbf{u}_\delta\|_{H^1}^2),$$

where the constant C depends on $\|\mathbf{f}\|_{L^\infty}$, $\|\mathbf{g}\|_{L^\infty}$, μ, $\tilde{\mu}$, M, γ and Ω, but not on δ.

Remark 2.12. *The uniform estimates (2.41) is the key estimate when proving Lemma 2.11.*

Recalling that Lemma 2.9 is true for any $s \in (1, \alpha + 1 - \alpha/\gamma]$, we write $s = 1 + \varepsilon$, where ε will be chosen small enough later on, and use Lemma 2.9–Lemma 2.11 to infer that

$$\begin{aligned} A &\leq C\|\mathbf{u}_\delta\|_{H^1}^2 \Big(1 + \|\mathbf{u}_\delta\|_{H^1}^2 + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^1} + \|P_\delta\|_{L^1}\Big) \\ &\leq C A^{\frac{\gamma-1}{2(\alpha\gamma+\gamma-2)}} \Big(1 + A^{\frac{\gamma-1}{2(\alpha\gamma+\gamma-2)}} + A^{\frac{\gamma s-1}{\gamma\alpha+\gamma-2}\cdot\frac{1}{1+\varepsilon}}\Big) \\ &\leq C\Big(1 + A^{\frac{3(\gamma-1)}{2(\gamma\alpha+\gamma-2)} + O(\varepsilon)}\Big). \end{aligned} \tag{2.45}$$

Since (2.45) remains valid for any $\alpha \in (0,1)$, if we write $\alpha = 1 - \sigma$ with $0 < \sigma < 1$, then

$$\gamma > \frac{1}{1-2\sigma} \qquad \Rightarrow \qquad \frac{3(\gamma-1)}{2(\gamma\alpha+\gamma-2)} < 1,$$

where ε and σ can be arbitrary small. So, by our choice of the parameters ε and σ, the exponent $\frac{3(\gamma-1)}{2(\gamma\alpha+\gamma-2)} + O(\varepsilon)$ can be made less than 1, and therefore we conclude by (2.45) that

$$A \leq C,$$

which immediately implies that

$$\begin{aligned} &\|\mathbf{u}_\delta\|_{H^1} \leq C, \\ &\|p_\delta\|_{L^r} + \|\rho_\delta|\mathbf{u}_\delta|^2\|_{L^r} \leq C, \\ &\|\rho_\delta \mathbf{u}_\delta\|_{L^r}^r = \int_\Omega (\rho_\delta^r \mathbf{u}_\delta^{2r})^{1/2} (\rho_\delta^r)^{1/2} \leq C\|\rho_\delta^r \mathbf{u}_\delta^{2r}\|_{L^1}^{1/2} \|\rho_\delta^r\|_{L^1}^{1/2} \leq C, \\ &\delta \int_\Omega \rho_\delta^6 dx \leq C\delta^{\frac{\gamma(s-1)}{6+\gamma(s-1)}} \Big(\int_\Omega \delta \rho_\delta^{6+\gamma(s-1)} dx\Big)^{\frac{6}{6+\gamma(s-1)}} \leq C\delta^{\frac{\gamma(s-1)}{6+\gamma(s-1)}}. \end{aligned}$$

Therefore, we obtain Theorem 2.8.

3 Proof of Theorem 1.2

In this section we will take to the limit as $\delta \to 0$ for the approximate problem (2.35)–(2.37) to obtain a weak solution of (1.1)–(1.3) for any $\gamma > 1$. The main task for completing the proof of Theorem 1.2 is to get the strong convergence of ρ_δ to ρ in $L^1(\Omega)$, and this will be fulfilled in several steps. In fact, we have already had all the necessary estimates, and we can then employ a process very similar to that used in [15, 12] to get the strong convergence of ρ_δ in $L^1(\Omega)$. Hence, we omit the details of the proof and only describe the main steps of the proof here.

First, the following preliminary lemma will be used in the proof of Theorem 1.2.

Lemma 3.1. *([12]) Let $1 < p_1, p_2, p < \infty$, $p \le p_1$ and Ω be a bounded domain in $\mathbb{R}^3$. Suppose that*

$$\begin{aligned} f_n &\rightharpoonup f \quad \text{weakly in } L^{p_1}(\Omega), \\ g_n &\to g \quad \text{strongly in } L^{p_2}(\Omega), \end{aligned}$$

and

$$f_n g_n \ \text{ are uniformly bounded in } L^p(\Omega).$$

Then there is a subsequence of $f_n g_n$ (still denoted by $f_n g_n$), such that

$$f_n g_n \rightharpoonup fg \quad \text{weakly in } \ L^p(\Omega).$$

Remark 3.1 *We point out here that $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ is not needed in Lemma 3.1.*

By Theorem 2.8, and the compact imbedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$, $p \in [1, 6)$, we have the following limits:

$$\begin{aligned} &\delta\rho_\delta^6 \to 0 && \text{in } \mathcal{D}'(\Omega), \\ &\mathbf{u}_\delta \rightharpoonup \mathbf{u} && \text{weakly in } H_0^1(\Omega), \\ &\mathbf{u}_\delta \to \mathbf{u} && \text{strongly in } L^p(\Omega), \quad 1 \le p < 6, \\ &\rho_\delta \rightharpoonup \rho && \text{weakly in } L^{\gamma r}(\Omega), \\ &\rho_\delta^\gamma \rightharpoonup \overline{\rho^\gamma} && \text{weakly in } L^r(\Omega). \end{aligned} \tag{3.46}$$

It is easy to see that Theorem 2.8, together with Lemma 3.1 and (3.46), yields that

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup \rho\mathbf{u} \quad \text{and} \quad \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup \rho\mathbf{u} \otimes \mathbf{u} \quad \text{in } L^r(\Omega). \tag{3.47}$$

Letting $\delta \to 0$ in (2.35) and (2.37), applying (3.46) and (3.47), we find that the weak limit $(\rho, \mathbf{u})$ of $(\rho_\delta, \mathbf{u}_\delta)$ satisfies

$$\begin{aligned} &\operatorname{div}(\rho\mathbf{u}) = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ &\operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + a\nabla\overline{\rho^\gamma} - \mu\Delta\mathbf{u} - \tilde{\mu}\nabla\operatorname{div}\mathbf{u} = \rho\mathbf{f} \quad \text{in } \mathcal{D}'(\Omega); \\ &\operatorname{div}[\overline{b(\rho_\delta)\mathbf{u}_\delta}] + \overline{[b'(\rho_\delta)\rho_\delta - b(\rho_\delta)]\operatorname{div}\mathbf{u}_\delta} = 0 \quad \text{in } \mathcal{D}'(\Omega), \\ &\int_\Omega \left[\mu|\nabla\mathbf{u}|^2 + \tilde{\mu}|\operatorname{div}\mathbf{u}|^2\right] dx \le \int_\Omega \rho\mathbf{f} \cdot \mathbf{u} + \mathbf{g} \cdot \mathbf{u} dx. \end{aligned}$$

As the next step, we need to prove an identity (i.e. Lemma 3.2 below) for the so-called effective viscous flux defined by

$$\tilde{H}_\delta := a\rho_\delta^\gamma - (\mu + \tilde{\mu})\operatorname{div}\mathbf{u}_\delta \rightharpoonup \tilde{H} := a\overline{\rho^\gamma} - (\mu + \tilde{\mu})\operatorname{div}\mathbf{u}, \qquad \text{as } \delta \to 0.$$

Lemma 3.2. *([15]) For any $\phi \in C_0^\infty(\Omega)$, we have*

$$\lim_{\delta\to 0}\int_\Omega \phi(x)\tilde{H}_\delta T_k(\rho_\delta)dx = \int_\Omega \phi(x)\tilde{H}\overline{T_k(\rho)}dx.$$

where $T_k(t)$ is the cut-off function:

$$T_k(t) := kT\Big(\frac{t}{k}\Big), \quad T(t) := \begin{cases} t, & t \le 1, \\ 2, & t \ge 3 \end{cases} \in C^\infty(\mathbb{R}), \quad \textit{concave.}$$

Remark 3.2. *By using the density argument, one can actually take $\phi(x) \equiv 1$.*

Lemma 3.3. (Control of the oscillation of the density) *We have*

$$\overline{\lim_{\delta\to 0}}\|T_k(\rho_\delta) - T_k(\rho)\|_{L^{\gamma+1}} \le C,$$

where the constant C is independent of k.

Lemma 3.4. (Renormalized continuity equation) *The weak limit $(\rho, \mathbf{u})$ is a renormalized solution of (1.1), i.e., (ρ, u) satisfies*

$$\mathrm{div}[b(\rho)\mathbf{u}] + [b'(\rho)\rho - b(\rho)]\mathrm{div}\mathbf{u} = 0 \quad \textit{in } \mathcal{D}'(\Omega)$$

for any $b \in C^1(\mathbb{R})$, $b'(z) = 0$ for sufficiently large z.

Now, introducing a family of functions

$$L_k(z) = \begin{cases} z\log z, & 0 \le z \le k \\ z\log k + z\int_k^z \frac{T_k(s)}{s^2}ds, & z \ge k \end{cases} \in C^1(\mathbb{R}_+) \cap C^0[0,\infty),$$

and making use of Lemmas 3.5 and 3.6, we argue, in the same manner as in [1], to conclude that

$$\lim_{\delta\to 0}\|\rho_\delta - \rho\|_{L^1} = 0,$$

which, by (3.46) and the interpolation theory, implies that

$$\rho_\delta \to \rho \quad \text{strongly in } \ L^p(\Omega), \qquad \forall\, 1 \le p < \gamma r.$$

Consequently, we have

$$\overline{\rho^\gamma} = \rho^\gamma, \qquad a.e.$$

Thus, the proof of Theorem 1.2 is complete.

References

[1] J. Březina and A.Novotný: On weak solutions of steady Navier-Stokes equations for monoatomic gas, *Comment. Math. Univ. Carolinae* **49** (2008), 611-632.

[2] E. Feireisl: *Dynamics of Viscous Compressible Fluids*, Oxford Science Publ., Clarendon Press, Oxford, 2003.

[3] E. Feireisl: On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not integrable, *Comment. Math. Univ. Carolinae* **42** (2001), 83-98.

[4] E. Feireisl and A. Novotný and H. Petzeltová: On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.* **3** (2001), 358-392.

[5] J. Frehse, S. Goj and M. Steinhauer: L^p-estimates for the Navier-Stokes equations for steady compressible flow, *Manuscripta Math.* **116** (2005), 265-275.

[6] J. Frehse, M. Steinhauer, and W. Weigant: On stationary solutions for 2-D viscous compressible isothermal Navier-Stokes equations, *J. Math. Fluid Mech.*, **13** (2010), 55-63.

[7] J. Frehse, M. Steinhauer, W. Weigant: The Dirichlet problem for viscous compressible isothermal Navier-Stokes equations in two-Dimensions, *Arch. Rational Mech. Anal.* **198** (2010), 1-12.

[8] J. Frehse, M. Steinhauer, W. Weigant: The Dirichlet problem for steady viscous compressible flow in 3-D, *J. Math. Pures Appl.* Doi:10.1016/j.matpur.2009.06.005.

[9] D. Gilbarg and N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, 2nd Edition, Springer-Verlag, 1983.

[10] S. Jiang and P. Zhang: Global spherically symmetry solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phy.* **215** (2001), 559-581.

[11] S. Jiang and P. Zhang: Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic fluids, *J. Math. Pure Appl.* **82** (2003), 949-973.

[12] S. Jiang and C.H. Zhou: Existence of weak solutions to the three-dimensional steady compressible Navier-Stokes equations, *Ann.I.H. Poincaré Analyse non linéaire* **28** (2011), 485-498

[13] P.L. Lions: *Mathematical Topics in Fluid Mechanics, Vol.II, Compressible Models.* Clarendon Press, Oxford, 1998.

[14] P.B. Mucha and M. Pokorný: Weak solutions to equations of steady compressible heat conducting fluids, *Math. Models Meth. Appl. Sci.* **20** (2010), 785-813.

[15] A. Novotný and I. Straškraba: *Introduction to the Mathematical Theory of Compressible Flow*, Oxford Univ. Press, Oxiford, 2004.

[16] P.I. Plotnikov and J. Sokolowski: Concentrations of solutions to time-discretizied compressible Navier-Stokes equations, *Comm. Math. Phy.* **258** (2005), 567-608.